Leslie Matrices

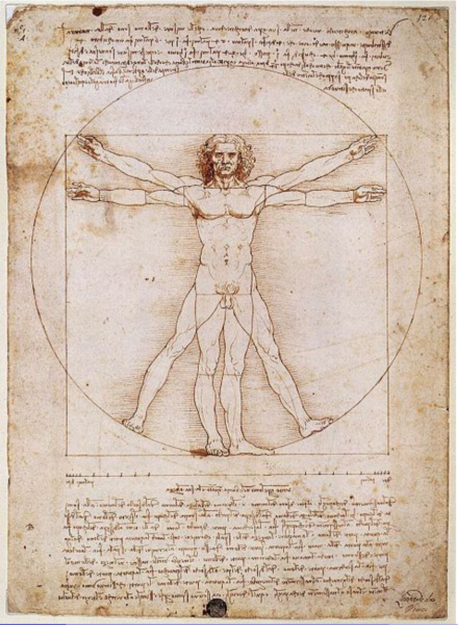
Modelling Age Structured Populations with Eigenvalues

Matthew Roughan
matthew.roughan@adelaide.edu.au

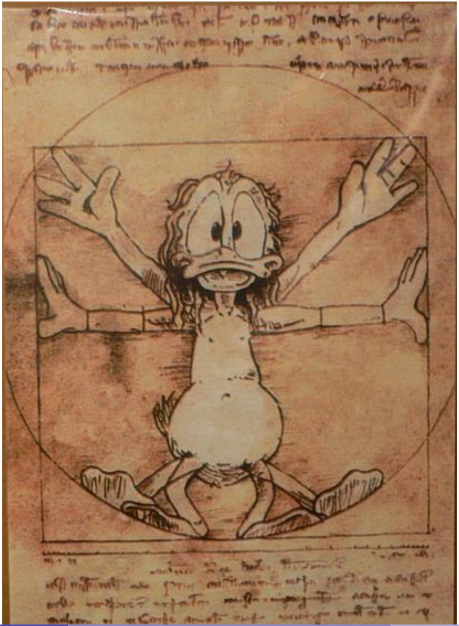
School of Mathematical Sciences
University of Adelaide

March 20, 2014

Maths as an Art



Maths as an Art



Maths as an Art

- Engineers and Scientist see Maths as a tool
  - Like a hammer, you get it out when you need it, and put it away when you don't
  - You don't think too hard about how to use a hammer, you just hit things with it
  - Some people build better hammers, but that's their problem, not mine
- I see Maths more like an art
  - Its a living corpus of work
  - If you are going to use it, you need to understand the loose edges
  - Everyone who uses Maths should be making it better

Population Models

There are lots of models of populations:

- Exponential growth
- Logistic growth
- Lotka-Volterra (predator-prey)
- Stochastic models: birth and death processes

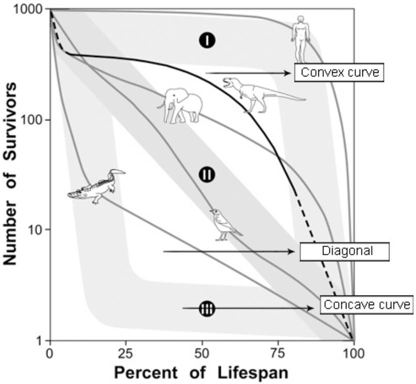
Most of them assume the population is homogeneous, but real populations have structure, e.g.,

- Male/female
- Geography
- Different ages

Ageing populations

The distribution of ages matters

- death rate can change with age
- birth rate can change with age



http://amrita.vlab.co.in/?sub=3&brch=65&sim=183&cnt=1

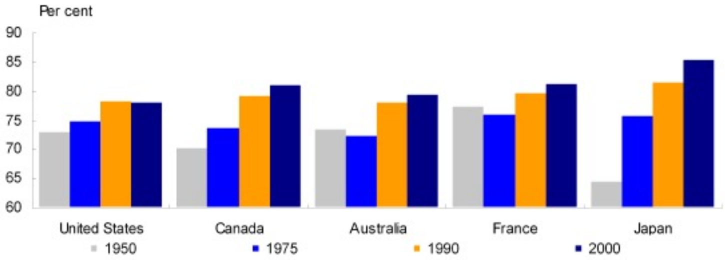
Example 1: Australian Demographics

Governments need to predict populations in different age categories in order to plan:

- Schools (how many children will there be?)
- Pensions (how many retired people will there be?)

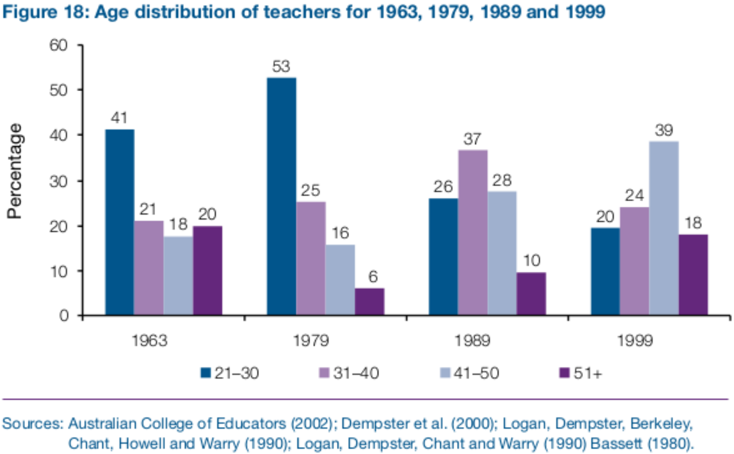
Australia has an "ageing" population.

Proportion of population over 15.



http://demographics.treasury.gov.au/content/_download/australias_demographic_challenges/html/adc-04.asp

Example 2: Australian Teachers



"Australia's Teachers: Australia's Future", Chapter 5, pp.53–64, DEST, Committee for the Review of Teaching and Teacher Education, 2003, ISBN 1 877032 80 8.

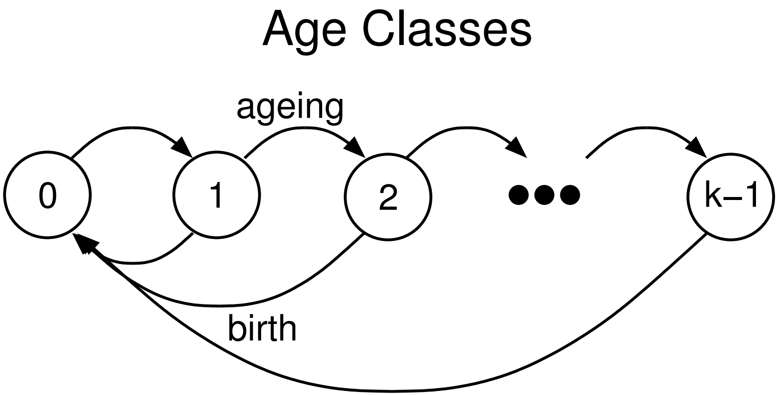
Example 3: Weed Killers

Imagine you want to control a weed (or other pest) and you have two choices of weedicide

1. is extremely effective, but only kills mature plants
2. is less effective, but kills germinating seeds

which is better?

The Model



- Age specific survival rate governs ageing, from class i to $i+1$.
- Age specific fecundity (per capita birth rate) governs births, but all births start in age category 0

Terminology

Each time step, from $t \to t+1$, individuals age and potential die, and/or give birth:

- **survival rate**: s_i is the proportion of individuals from Age Class i that survive to $i+1$.
- **fecundity**: f_i is the proportion of individuals from Age Class i who give birth to new individuals in Age Class 0.
- **population**: at time step t is kept in the vector $\mathbf{n}_t$.

The above often only makes sense if we model female populations (as males don't give birth).

The Leslie Matrix: Definition

The equation for one time step of the model as

$$\begin{pmatrix} n_{t+1}(0) \\ n_{t+1}(1) \\ \vdots \\ n_{t+1}(k-1) \end{pmatrix} = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & \dots & f_{k-1} \\ s_0 & 0 & 0 & 0 & \dots & 0 \\ 0 & s_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & s_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \dots & 0 \\ 0 & 0 & 0 & \dots & s_{k-2} & 0 \end{pmatrix} \begin{pmatrix} n_t(0) \\ n_t(1) \\ \vdots \\ n_t(k-1) \end{pmatrix}$$

or more succinctly as

$$\mathbf{n}_{t+1} = L\mathbf{n}_t$$

where L is called the **Leslie Matrix.**

The Leslie Matrix Equation

Simple extrapolation of the equation

$$\mathbf{n}_{t+1} = L\mathbf{n}_t$$

from the first time step, where the population is $\mathbf{n}_0$ gives

$$\mathbf{n}_t = L^t\mathbf{n}_0$$

so we can calculate future populations, just by taking powers of the Leslie matrix.

Let's Play

Login

- Username:
- Password:

Open Internet Explorer (not Firefox), and go to the following URL:

`http://bandicoot.maths.adelaide.edu.au/Leslie_matrix/leslie.cgi`

What you should see

Leslie Matrix Calcul... | MathJax Test Page | MathJax Example Pa... | Dynamic Preview of... | Leslie Matrix Calcul...

bandicoot.maths.adelaide.edu.au/Leslie_matrix/leslie.cgi

apache ServerAlias

Most Visited | Google | Adelaide Maths | whitepages.com.au ... | Matthew Roughan: ...

Leslie Matrices:

LESLIE MATRICES:

Leslie Matrices are used to model growth (and decline) of age-structured populations. For instance, in Australia it is widely reported that we have an aging population. How do demographers know this?

In the model named after Patrick H. Leslie (1945), we have N age classes, and we record how many individuals are in each. Then, each time period, people either age (and consequently move to the next age class), or die. The **survival rate** for each age class describes the proportion of the population that moves onto the next age class. New individuals can also be born, and the **birth rate** , or fecundity describes the rate per capita of births arising from each age category.

Given each of these parameters, we can model the evolution of a single time step with the equation

$$\mathbf{n}_{t+1} = L\mathbf{n}_t,$$

where $\mathbf{n}_t$ is a vector of the populations in each age class at time t and L is the Leslie Matrix.

CHOOSE MATRIX PARAMETERS:

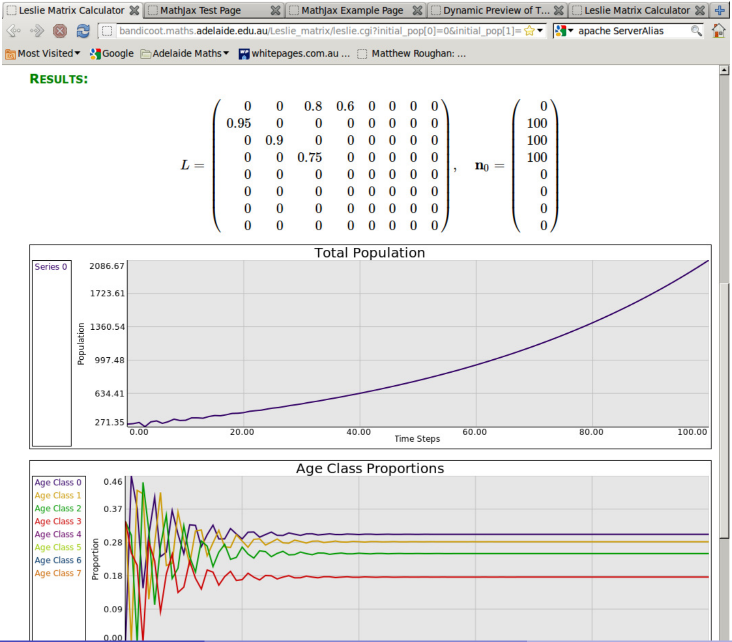
Fill in the fields below. All values must be ≥ 0. Survival rates must also be ≤ 1. Invalid numbers will be truncated, and all will be rounded to three decimal places. Obviously there is a maximum of 8 age classes here, but you don't need to use them all. Set 0 to the survival rate of one age class, and all those above will die out.

Leslie Model Parameters

Age Class	0	1	2	3	4	5	6	7
Initial Population	1	0	0	0	0	0	0	0
Birth Rate	0	0	0	0	0	0	0	0
Survival Rate	0	0	0	0	0	0	0	0

Submit Leslie Matrix it may take a couple of seconds to generate a new set of results.

Results



$$L = \begin{pmatrix} 0 & 0 & 0.8 & 0.6 & 0 & 0 & 0 & 0 \\ 0.95 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{n}_0 = \begin{pmatrix} 0 \\ 100 \\ 100 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

What we should see

- The model parameters (survival rate, and fecundity) play a big role in determining whether the population lives or dies.
- The starting population isn't so important.
  - Growth or decay aren't determined by starting populations.
  - The final proportions of each Age Class don't depend on the starting proportions
- In many cases it's quite hard to guess whether a population will grow or die.

What you may have noticed

- The calculator also reports two extra results:
  - The first eigenvalue, which we will denote λ_1
  - Its corresponding eigenvector
- You may have noticed
  - Growth and decay are linked to the eigenvalue:
    If $\lambda_1 > 1$ you get growth
    If $\lambda_1 < 1$ you get decay
  - The final proportions of each Age Class match the eigenvector

Eigenvalues and Eigenvectors

Definition: Take a square $n \times n$ matrix A, then a non-zero vector in $\mathbf{x} \in \boldsymbol{R}^n$ is called an **eigenvector** if and only if it satisfies

$$A\mathbf{x} = \lambda \mathbf{x}$$

for some scalar λ, which is called an **eigenvalue** of A.

$\mathbf{x}$ is said to be the eigenvector corresponding to λ.

Why does it work?

The other session will talk some more about eigenvalues, but the approximate view here is

$$\mathbf{n}_t = L^t \mathbf{n}_0 \simeq \gamma\, \lambda_1^t\, \mathbf{x}_1$$

for large t, where λ_1 is the largest eigenvalue of L, and $\mathbf{x}_1$ is its corresponding eigenvector.

Conclusion

Modelling is all about tractable vs realism tradeoffs

- Maths models for growth are somewhat limited
  - need to account for age
- The Leslie model provides a very simple way to do so
- Mathematical analysis can be used to understand its behaviour
- But the Leslie model still has limitations
  - no migration
  - it's linear
  - only one species